

# Summary of ‘On the stability of sequential Monte Carlo methods in high dimensions’

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## 1 Context

In recent years there has been an explosion of data in a wide variety of applied disciplines such as finance and genetics. As a result, the primary job of many statisticians has been to develop advanced and complex stochastic models in order to capture the underlying physical phenomenon. In many of these statistical models one is interested to estimate a variety of parameters which describe the data, for example in finance the ‘volatility’ of a price in a financial market. In such scenarios this is categorized by evaluating an integral, or if one prefers a ‘theoretical average’, in very high-dimensions. In practice, one does not know the analytic value due to the high-dimensional nature of the integral. As a result, this has led to a substantial literature on numerical approximation of integrals, which can be roughly divided into stochastic and deterministic methods.

It is widely believed that stochastic numerical integration methods out-perform, in dimensions bigger than 3, deterministic methods. In general, statisticians and probabilists focus upon stochastic ideas, in particular Monte Carlo approaches. Monte Carlo methods are believed (wrongly) to always break the curse of dimensionality and accurately estimate integrals in very high-dimensions. This is true in some scenarios, but there are instances where the methodology has very poor performance and yields very inaccurate answers. That is, if the dimension of the problem is  $d$ , then it is thought that the method can work well if, for some good behavior the cost is polynomial in  $d$ , and work poorly if the cost is exponential in  $d$ . Roughly, the idea of Monte Carlo is to rewrite an integral as a theoretical average of a function with respect to a probability model. Then to simulate samples from this probability distribution and approximate the integral by the average of the function evaluated at the simulated samples. In practice, in high-dimensions, one needs to use advanced methods such as Markov chain Monte Carlo (MCMC).

## 2 Summary

It is important to evaluate the cost of simulation methods. In a landmark article [4] established not only the cost, in  $d$ , of the most used MCMC method, but also some optimality results associated to it. The article showed that the computational cost was only polynomial in  $d$ , although the major result of the article went far beyond this point. In many empirical studies it has been found that Sequential Monte Carlo (SMC) techniques out-perform MCMC methods and as yet, no study of the computational cost has been undertaken. However a distinctly negative result of [3] showed that importance sampling (the basis of SMC) has an exponential cost in  $d$ , leading to many researchers avoiding SMC. It should be noted that a study of SMC in high-dimensions is important as it applies to many problems beyond MCMC, such as ‘sequential inference’ (filtering) - the updating of estimates as new data arrive.

In recent work [1, 2], we have shown that a particular SMC method has a cost in the dimension that is only polynomial in  $d$ , albeit the same order as some MCMC algorithms. These are the first positive results in the area and help to confirm that SMC is viable option in high-dimensional problems. These mathematical results also shed some light on the phenomena observed by practitioners. Unfortunately, at present, they do not extend to filtering, and this is perhaps the major research challenge in the field: to find an SMC algorithm for filtering, that has cost that has a polynomial cost in  $d$  for a wide class of problems.

Below, one of the theorems in the article [1] is given. What it tells us is that the SMC estimates of integrals have a mathematically meaningful limit as  $d$  grows, when the computational cost is polynomial in  $d$ .

**Theorem 2.1.** *Assume (A1-2) with  $g \in \mathcal{L}_{V^r}$  for some  $r \in [0, \frac{1}{2})$ . Then for any  $1 \leq \varrho < \infty$  there exists a constant  $M = M(\varrho) < \infty$  such that for any  $N \geq 1$ ,  $\varphi \in \mathcal{C}_b(\mathbb{R})$*

$$\lim_{d \rightarrow \infty} \left\| \sum_{i=1}^N \frac{w_d(X_{0:d-1}^i)}{\sum_{l=1}^N w_d(X_{0:d-1}^l)} \varphi(X_{d,1}^i) - \pi(\varphi) \right\|_e \leq \frac{M(\varrho) \|\varphi\|_\infty}{\sqrt{N}} \left[ e^{\frac{\sigma_z^2}{2} \varrho(\varrho-1)} + 1 \right]^{1/\varrho}.$$

## References

- [1] BESKOS, A., CRISAN, D., JASRA, A. (2011). On the stability of sequential Monte Carlo methods in high dimensions. Under Submission *Ann. Appl. Probab.*.
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- [4] ROBERTS, G. O., GELMAN, A. & GILKS W. R. (1997). Weak convergence and optimal scaling of random walk Metropolis algorithms. *Ann. Appl. Probab.*, **7**, 110–120