

Scalar curvature flow method and prescribed scalar curvature problem

We, jointly with Dr. Chen Xuezhong from Nanjing University of China, has established a new method (namely the scalar curvature flow method) to handle the long-standing problem: is that possible to continuously deform the rough sphere to the perfect sphere? Mathematically there are several ways to measure the roughness. The scalar curvature measurement is the weakest one. With the scalar curvature measurement, the problem is commonly called the prescribing scalar curvature problem.

Their paper “The Scalar curvature flow on S^n - perturbation theorem revisited” has been published by *Inventiones* (187, No 2, pp395-506 with an erratum pp 506-509), one of the most prestigious journals in the mathematical community.

The prescribing scalar curvature problem has existed for more than a half century. Many great mathematicians have made important contributions. Such problem can be converted to finding some positive solutions of elliptic type partial differential equations. Such equation is always invariant under a non-compact group. The bubbling phenomena (very sharp needle) can happen. It is known among mathematicians that above two difficulties can only occur in the particular manifolds, namely the perfect sphere case.

In the seventies, French mathematician, T. Aubin, proved some interesting result, saying that if the rough sphere is relatively flat, up to some second lowest tone, it can be deformed to the perfect sphere without changing its angle measure.

In the early nineties, A. Chang and P. Yang, now both are Professors of Princeton University, had rechecked this problem and found that if the rough sphere is sufficiently close to the perfect sphere and has no degenerate sharp needle with extra technical assumptions, the deformation is possible. Their argument is based on the perturbation method and the calculus of variation.

In the past ten years, the most significant mathematical achievement is the reconfirmation of the famous Poincaré conjecture. We all know that the proof is through the powerful Hamilton’s Ricci flow which was introduced near thirty years ago. Its success in Poincaré conjecture stimulates many people to re-examine the Ricci flow and apply it to other important problems.

In the last several years, we adopted this flow method to the prescribing scalar curvature problem. We obtained in certain sense the best possible conclusion under the same assumption as ones by Aubin and Chang-Yang. Basically we show that, with Aubin’s relative flatness assumption plus Chang-Yang’s topological assumptions, the problem is solvable. The important observation in this work is that the relative flatness given by Aubin is in fact to ensure the flow can have only one sharp needle which makes the blow-up analysis much easier.

The other tool we used in this work is the infinitely dimensional Morse theory. As it is well known, the difficult part in this theory is to control the energy level since the global existence depends on the initial energy level.