NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

ADVANCED PLACEMENT TEST
(SAMPLE)

MA1102R CALCULUS

MMM-YYYY — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. **Write your name here:** _____
2. This paper contains a total of **SIX (6)** questions and comprises **NINETEEN (19)** printed pages, including this page.
3. This is a **CLOSED BOOK** test. No list of formulas is provided and helpsheets are disallowed.
4. Only non-programmable and non-graphing calculators without remote communication function may be used. However, you should lay out systematically the various steps in your calculations.
5. Candidates must answer **ALL** 6 questions.
6. **Write your solutions in the spaces provided below the questions in this test paper. Submit this test paper at the end of the test period.**

Question 1 [16 marks]

(a) Find the volume of the largest right circular cone that can be inscribed in a sphere of radius $3cm$.

(b) Let a, b, c and d be positive integers. Prove that

$$\frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)}{abcd} \geq 16.$$

Solution:

(*More space for solution to Question 1.*)

(*More space for solution to Question 1.*)

Question 2 [16 marks]

(a) Evaluate

$$\lim_{x\to\ 0}\ (\frac{1+2^x+3^x}{3})^{1/x}.$$

(b) Use an appropriate Riemann sum to evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}(\frac{k^2}{n^3}+\frac{\sqrt{k}}{n^{3/2}}).$$

Solution:

(*More space for solution to Question 2.*)

(*More space for solution to Question 2.*)

Question 3 [16 marks]

(a) Given that

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos x+\sin\,x}{1+x^2}\,dx=\pi e^{-1},$$

evaluate

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos x}{1+(1-x)^2}\,dx.$$

(b) Let f be a differentiable function on $[0,1]$ such that $f(0)=0$ and $f(1)=1$. If the derivative f' of f is also continuous on $[0,1]$, prove that

$$\int_0^1|f'(x)-f(x)|\,dx\geq\frac{1}{e}.$$

[Hint: Let $h(x)=e^{-x}f(x)$.]

Solution:

(*More space for solution to Question 3.*)

(*More space for solution to Question 3.*)

Question 4 [16 marks]

(a) Let f be a differentiable function on $[0,1]$ such that $f(1) = 1$. If the derivative of f is also continuous on $[0,1]$, evaluate

$$\lim_{y\to\infty} \int_0^1 yx^y f(x)\,dx.$$

(b) Let f be a differentiable function on $(-\infty,\infty)$ and let f' be the derivative of f. If $f'(x) \neq 0$ for all $-\infty < x < \infty$ and $f'(0) = 1$, prove that $f'(x) > 0$ for all $-\infty < x < \infty$.

Solution:

(*More space for solution to Question 4.*)

(*More space for solution to Question 4.*)

Question 5 [16 marks]
Let f be a differentiable function on $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$. Prove that there exist $x_1, x_2 \in [0, 1]$ such that

$$\frac{1}{f'(x_1)} + \frac{1}{f'(x_2)} = 2,$$

where f' denotes the derivative of f.

Solution:

(*More space for solution to Question 5.*)

(*More space for solution to Question 5.*)

Question 6 [20 marks]

(a) Let $y(x)$ be the solution of the initial value problem

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}, \qquad y(\pi) = 0.$$

Find the value of $y(2\pi)$.

(b) A tank has a volume of 30 litres. Initially, it has 10 litres of salt solution and the concentration of salt in the solution is 2 kg per litre. A salt solution of concentration 0.5 kg per litre is being poured into the tank at a constant rate of 2 litres per minute. The well-mixed solution is constantly being pumped out of the tank at a rate of 1 litre per minute. Find the amount of salt (in kg) in the tank at the time when the tank becomes full, that is, it has 30 litres of salt solution.

Solution:

(*More space for solution to Question 6.*)

(*More space for solution to Question 6.*)

End of Test